ON THE ASYMPTOTIC SOLUTION OF THE EQUATIONS OF MOTION OF A GYROSCOPIC COMPASS

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Certain asymptotic solutions of the equations of motion of a two-rotor gyrocompass are discussed in the case where the stiffness of the characteristic of the restoring moment due to the spring coupling between the gyroscopes is large. Use is made of the results obtained in [1-3].

1. The differential equations of motion of a two-rotor gyrocompass obtained in [3] are of the form

$$\alpha^{\cdot} - \frac{\nu^2}{u\cos\varphi}\beta - \Omega\tan\varphi\delta = 0, \qquad \gamma^{\cdot} + \frac{p^2}{\nu^2}u\sin\varphi\delta + \Omega\beta = 0$$
$$\beta + u\cos\varphi\alpha - \Omega\gamma = 0, \qquad \delta^{\cdot} - \frac{\nu^2}{u\sin\varphi}\gamma + \Omega\cot\varphi\alpha = 0 \tag{1.1}$$

This motion does not possess the properties of a Geckeler-Anschütz gyrocompass.

The symbols used in Equation (1.1) are defined in [3], Equation (2.3). The only difference is that the required functions are denoted by α, β, γ and δ. Equations (1.1) can be written down in the form of the following two equations in α and δ:

$$\alpha^{\cdot\cdot} + (\nu^2 - \Omega^2)\alpha = 2\Omega\tan\varphi\delta^{\cdot} + \Omega\tan\varphi\delta$$
$$\varepsilon\delta^{\cdot\cdot} + (\nu^2 - \varepsilon\Omega^2)\delta = -2\varepsilon\Omega\cot\varphi\alpha^{\cdot} - \varepsilon\Omega\cot\varphi\alpha \qquad \left(\varepsilon = \frac{\nu^2}{p^2}\right) \tag{1.2}$$

It was shown in [3] that one can set

$$p = \frac{\sqrt{Pls}}{2B\sin\varphi} = \mathrm{const} \tag{1.3}$$

We recall that Pl is the pendulum moment of the gyrosphere, s is the stiffness of the characteristic of the restoring couple which depends on the spring coupling between the gyroscopes, B is the proper angular momentum of the rotor and ϕ is the latitude.

2. Let us assume that the stiffness s is chosen so large that the dimensionless parameter ϵ (which is defined by Equation (1.2)) is small as compared to unity. The quantities a and δ can then be written down in the form of the series

$$\alpha = \alpha_0 + \sum_{n=1}^{\infty} \varepsilon^n \alpha_n, \qquad \delta = \delta_0 + \sum_{n=1}^{\infty} \varepsilon^n \delta_n \tag{2.1}$$

Substituting Equation (2.1) into Equation (1.2) and equating coefficients of equal powers of ϵ, one can successively obtain equations from which the functions a_n and δ_n can be determined. In particular, a_0 and δ_0 are given by

$$\alpha_0^{\cdot\cdot} + (\nu^2 - \Omega^2)\,\alpha_0 = 0, \qquad \delta_0 = 0 \tag{2.2}$$

and, in general, in view of Equations (1.2), (1.3) and (2.1)

$$\alpha = \alpha_0 + O(s^{-1}), \qquad \delta = O(s^{-1}) \tag{2.3}$$

where the symbol $O(s^{-1})$ denotes all the terms of order s^{-1} or higher.

We shall look upon Equation (2.2) as an asymptotic representation (when $s \to \infty$, $\epsilon \to 0$) of the solutions of Equation (1.1).

We shall be mainly interested in the behavior of the gyrocompass during successive circulations of a ship. In such cases, one can set [3]

$$\Omega = -\mu\omega \sin \omega t \quad (\mu = v\,/\,Ru \cos \varphi,\ \omega = 2\pi/T) \tag{2.4}$$

where v is the velocity of the ship in circulation and T is the circulation period.

In view of Equation (2.4), the first of Equations (2.2) can be rewritten in the form

$$\alpha_0^{\cdot\cdot} + k(t)\,\alpha_0 = 0 \tag{2.5}$$

where

$$k(t) = \nu^2 (1 - m + m \cos 2\omega t) \qquad \left(m = \frac{\mu^2\omega^2}{2\nu^2}\right) \tag{2.6}$$

3. The series given by Equation (2.1) were introduced formally without proving their convergence. It may be assumed that these series will converge so long as $\epsilon < A$, where A is a positive constant. This condition can always be satisfied by choosing s to be sufficiently large.

In the case of variable or, in particular, periodic coefficients in Equation (2.5), this assumption must, of course, be verified. Moreover, one can give a simple mechanical interpretation to the transition to Equation (2.2).

In fact, by increasing the stiffness of the spring between the gyroscopes (and consequently the curvature s), we will deprive the system of one of its degrees of freedom, e.g. in the present case, the motions of the gyroscopes with respect to the axes of their housings, which are characterized by the coordinate δ. It follows that in the limiting case one should have $\delta = \delta \equiv 0$, and this leads to Equation (2.5).

We present now another method for obtaining Equations (2.2) and (2.3) which does not depend on the series given by Equation (2.1). Bearing in mind Equation (2.4), the second equation in (1.2) can be rewritten in the form

$$\delta^{\cdot\cdot} + \lambda^2\delta = -\tfrac{1}{2}\mu^2\omega^2\cos 2\omega t\delta + \mu\omega^2 \cot\varphi\cos\omega t\alpha + 2\mu\omega\cot\varphi\sin\omega t\alpha^{\cdot} \tag{3.1}$$

In this equation

$$\lambda^2 = p^2 - \tfrac{1}{2}\mu^2\omega^2 \tag{3.2}$$

Assuming that $p^2 - 1/2\,\mu^2\omega^2 > 0$, Equation (3.1) leads to the integro-differential equation

$$\delta = C_1\cos(\lambda t + \psi) + \frac{1}{\lambda}\int_0^t K_1(t,\xi)\,\delta(\xi)\,d\xi + \frac{1}{\lambda}\int_0^t K_2(t,\xi)\,\alpha(\xi)\,d\xi + \frac{1}{\lambda}\int_0^t K_3(t,\xi)\,\alpha^{\cdot}(\xi)\,d\xi \tag{3.3}$$

where C_1 and ψ are constants and, moreover,

$$\begin{aligned} K_1(t,\xi) &= -\tfrac{1}{2}\mu^2\omega^2\sin\lambda(t-\xi)\cos 2\omega\xi \\ K_2(t,\xi) &= \mu\omega^2\cot\varphi\sin\lambda(t-\xi)\cos\omega\xi \\ K_3(t,\xi) &= 2\mu\omega\cot\varphi\sin\lambda(t-\xi)\sin\omega\xi \end{aligned} \tag{3.4}$$

Consider a certain interval $(0,\ t^*)$ of the values of t (for example, the interval $0,\ \pi/\omega$, which corresponds to the semicirculation of the ship).

Suppose that M_1, M_2 and M_3 are the upper limits of $|\delta(t)|$, $|\alpha(t)|$, $|\alpha^{\cdot}(t)|$, respectively, in the interval $(0, t^*)$. Equations (3.3) and (3.4) give

$$|\delta(t) - C_1 \cos(\lambda t + \psi)| < \frac{1}{\lambda}\left(\frac{1}{2} M_1 \mu^2 \omega^2 + M_2 \mu \omega^2 \operatorname{ctg} \varphi + 2 M_3 \mu \omega \operatorname{ctg} \varphi\right) \tag{3.5}$$

It follows from the condition given by Equation (3.5) that for any t in the interval $(0, t^*)$ one can always find sufficiently large λ (or, in view of Equation (3.2), sufficiently large s) for the function $\delta(t)$ to be as close as desired to $C_1 \cos(\lambda t + \psi)$. The function $C_1 \cos(\lambda t + \psi)$ will, in fact, be the asymptotic representation of δ for large values of the parameter s.

Substituting this value into the first of the equations in (1.2) we obtain an equation for $\alpha(t)$ whose homogeneous part is

$$\alpha^{\cdot\cdot} + (\nu^2 - \Omega^2)\alpha = 0 \tag{3.6}$$

which is identical with Equation (2.2).

If one is concerned with specific initial conditions for δ, e.g. $\delta(0) = 0$, $\delta^{\cdot}(0) = h$, then one could similarly obtain the following estimate for the interval $(0, t^*)$:

$$|\delta(t)| < \frac{1}{\lambda}\left(h + \frac{1}{2} M_1 \mu^2 \omega^2 + M_2 \mu \omega^2 \operatorname{ctg} \varphi + 2 M_3 \mu \omega \operatorname{ctg} \varphi\right) \tag{3.7}$$

4. Let us now return to Equations (2.2) and (2.5). If $\Omega = \text{const}$, then it follows from Equation (2.2) that when

$$\Omega > \nu \tag{4.1}$$

Equation (2.2) has unstable solutions.

In the more interesting case, corresponding to the circulation of a ship, we must start with Equations (2.5) and (2.6).

From Equation (2.6) we have $k(t) > 0$ when $m < 1/2$, which together with Equation (2.4) leads to the inequality

$$v < \frac{T}{T_0} R u \cos \varphi \qquad \left(T_0 = \frac{2\pi}{\nu} = 2\pi \sqrt{\frac{R}{g}} \approx 84.4 \text{ min}\right) \tag{4.2}$$

Assuming that the condition $m < 1/2$ is satisfied, let us apply the Liapunov stability criterion to Equation (2.5). According to this criterion

$$\nu^2 \frac{\pi}{\omega} \int_0^{\pi/\omega} (1 - m + m \cos 2\omega t)\, dt \leqslant 4$$

We thus have the condition

$$T \leqslant \frac{2}{\pi} \frac{T_0}{\sqrt{1-m}} \tag{4.3}$$

in which the inequality sign applies for all the practically possible relations between T and T_0.

Equation (4.2) can thus be looked upon as a sufficient condition for the non-asymptotic stability of the system.

5. Let us now consider the conditions for instability. Substituting $\omega t = \tau$, Equation (2.5) can be rewritten in the form

$$\frac{d^2\alpha_0}{d\tau^2} + (a - 2q\cos 2\tau)\,\alpha_0 = 0 \quad \left(a = \frac{\nu^2}{\omega^2}(1-m),\ \ 2q = -m\frac{\nu^2}{\omega^2}\right) \tag{5.1}$$

Substituting $\alpha_0 = \alpha_0^{(0)} + q\alpha_0^{(1)} + q^2\alpha_0^{(2)} + \ldots$, for $\alpha_0^{(0)}$ we have

$$\frac{d^2\alpha_0^{(0)}}{d\tau^2} + a\alpha_0^{(0)} = 0 \tag{5.2}$$

whose solutions are unstable for $a < 0$, i.e. using Equations (2.4), (2.6) and (5.1)

$$v > \frac{T}{T_0}\sqrt{2}\,Ru\cos\varphi \tag{5.3}$$

However, the instability of solutions of Equation (5.2) does not, strictly speaking, lead to instability for Equation (5.1). In this connection let us consider the transcendental equation for the characteristic index κ of Equation (5.1), which is of the form [4]

$$\sin^2\frac{i\pi\kappa}{2} = \Delta(0)\sin^2\frac{\pi\sqrt{a}}{2} \qquad (i = \sqrt{-1}) \tag{5.4}$$

In this equation $\Delta(0)$ is the Hill determinant, which we shall calculate with the aid of the asymptotic formula of Tisserand [4]:

$$\Delta(0) = 1 + \pi\frac{q^2\cot\left(\frac{1}{2}\pi\sqrt{a}\right)}{\sqrt{a}\,(1-a)} + O(q^4) \tag{5.5}$$

Since, in practice, $T \ll T_0$, we find that, using Equations (2.6), (5.1) and the expansion

$$\cot z = \frac{1}{z} - \left(\frac{z}{3} + \frac{z^3}{45} + \cdots\right) \qquad (0 < |z| < \pi) \tag{5.6}$$

Equation (5.5) can be rewritten in the form

$$\Delta(0) \approx 1 - \frac{m^2}{2(m-1)}\left(\frac{\nu}{\omega}\right)^2 \tag{5.7}$$

In this equation we must assume $m \neq 1$, since in the opposite case it turns out that $a = 0$ and Equation (5.5) does not apply. When $m \neq 1$, the second term in Equation (5.7) is very small as compared with unity for all cases which are of interest in practice.

For example, when $\phi = 70°$, $v = 24$ knots and $T = 4$ min, we find that $\Delta(0) \approx 1 - 253 \times 10^{-5}$; when $\phi = 80°$, $v = 24$ knots, $T = 4$ min, we have $\Delta(0) \approx 1 - 654 \times 10^{-5}$.

In this case, we may assume that $\Delta(0) = 1$, and hence Equation (5.4) gives

$$\varkappa_{1,2} = \pm \frac{\nu}{\omega} \sqrt{1 - mi} \tag{5.8}$$

It follows that when $m > 1$, i.e. when the condition given by Equation (5.3) is satisfied, we are led to instability, since one of the values of κ is then positive.

Equations (4.2) and (5.3) impose rigid conditions on the circulation parameters. Thus, for example, when the circulation period T is equal to 4 min, instability occurs when $\phi = 70°$, $v > 20.6$ knots and when $\phi = 80°$, $v > 10.5$ knots.

6. The above theory can be used to study the behavior of a simple pendular single-rotor gyrocompass during the motion of the ship [1]. The equations for this compass can be obtained from the equations given in [2]. We thus have (see also [3])

$$\begin{gathered} H\alpha^{\cdot} + \frac{Pl}{g} V^{\cdot}\alpha - Pl\beta = \left(\frac{Pl}{g} V - H\right)\Omega \\ C\gamma^{\cdot\cdot} + Pl\gamma - \frac{Pl}{g} V\Omega\alpha = \frac{Pl}{g} V^{\cdot} - H^{\cdot} \\ H\left(\beta^{\cdot} + \frac{V}{R}\,\alpha\right) - \left(\frac{Pl}{g}\,V^{\cdot} - H^{\cdot}\right)\beta - \frac{Pl}{g} V\Omega\gamma = 0 \end{gathered} \tag{6.1}$$

In this equation, H is the proper angular momentum of the gyroscope rotor and C is the moment of inertia of the sensitive element relative to the North-South line.

We note that the last equation in (6.1) was constructed relative to the $z°$ axis of the Darboux trihedron oriented as in [2].

In single-rotor gyrocompasses H is usually constant. Let us assume in addition that $V \approx Ru \cos\phi + v_E$, so that Equation (6.1) leads to

$$\begin{gathered} H\alpha^{\cdot} + \frac{Pl}{g} v_E^{\cdot}\,\alpha - Pl\beta = \left[\frac{Pl}{g}\,(Ru\,\cos\varphi + v_E) - H\right]\Omega \\ H\left[\beta^{\cdot} + \left(u\,\cos\varphi + \frac{v_E}{R}\right)\alpha\right] - \frac{Pl}{g}\,v_E^{\cdot}\beta - \frac{Pl}{g}(Ru\cos\varphi + v_E)\,\Omega\gamma = 0 \\ C\gamma^{\cdot\cdot} + Pl\gamma - \frac{Pl}{g}\left(Ru\cos\varphi + v_E\right)\Omega\alpha = \frac{Pl}{g}\,v_E^{\cdot} \end{gathered} \tag{6.2}$$

If, in Equation (6.2), we neglect the East component of the ship's velocity v_E in comparison with $Ru \cos \phi$, together with the terms containing v_E, $\dot{v}_E$ and Ω on the left-hand sides, and substitute $\Omega = \dot{v}_N / Ru \cos \phi$ into the right-hand side of the first of the above equations, we obtain the well-known Geckeler-Bulgakov equations for a single-rotor compass [1]:

$$\beta^{\cdot} + u\cos\varphi\alpha = 0, \qquad C\gamma^{\cdot\cdot} + Pl\gamma = \frac{Pl}{g} v_E^{\cdot}$$

$$\alpha^{\cdot} - \frac{k^2}{u\cos\varphi}\beta = \left(\frac{k^2}{\nu^2} - 1\right)\frac{v_N^{\cdot}}{Ru\cos\varphi} \qquad \left(k = \sqrt{\frac{Pl\,u\cos\varphi}{H}}\right) \tag{6.3}$$

Consider now the equations given by (6.2), and assume that the ship is moving uniformly along a meridian.

Substituting $\Omega = \text{const}$, $v_E = v_E^{\cdot} \equiv 0$ into Equation (6.2), and bearing in mind Equation (6.3), we obtain the following characteristic equation for this system:

$$\lambda^4 + \frac{k^2}{C}\left(C + \frac{Pl}{k^2}\right)\lambda^2 + \frac{Pl}{C}\left(\frac{k}{\nu}\right)^4\left(\frac{\nu^4}{k^2} - \Omega^2\right) = 0 \tag{6.4}$$

If condition $k = \nu$ is satisfied at a given latitude, then Equation (6.4) will assume the form

$$\lambda^4 + \frac{\nu^2}{C}\left(C + \frac{Pl}{\nu^2}\right)\lambda^2 + \frac{Pl}{C}(\nu^2 - \Omega^2) = 0 \tag{6.5}$$

It follows that when $\Omega > \nu$, the solutions of the system given by Equation (6.2) will be unstable. The inequality $\Omega > \nu$ is clearly identical with the inequality (4.1) obtained earlier under similar conditions.

In studying the behavior of a single-rotor compass during circulations of the ship, one could use the method described in [3].

Assume that $H = (Pl/g)V$. Using this equality and setting $\alpha = (\alpha_1 Ru/V) \cos \phi$, we can rewrite Equation (6.1) in the form

$$\alpha_1^{\cdot} - \frac{\nu^2}{u\cos\varphi}\beta = 0,\ C\gamma^{\cdot\cdot} + Pl\gamma - \frac{Pl\,u\cos\varphi}{\nu^2}\Omega\alpha_1 = 0,\ \beta^{\cdot} + u\cos\varphi\alpha_1 - \Omega\gamma = 0 \tag{6.6}$$

In the case of circulation of the ship, Ω is determined by Equation (2.4). In order to obtain the characteristic equation we must have the fundamental matrix of solutions of Equation (6.6) at $t = T$. Let us substitute

$$\sin\omega t = \begin{cases} \dfrac{2}{\pi} - \dfrac{4}{\pi}\left(\dfrac{\cos 2\omega t}{3} + \ldots\right) & \left(0 \leqslant t \leqslant \dfrac{\pi}{\omega}\right) \\ -\dfrac{2}{\pi} + \dfrac{4}{\pi}\left(\dfrac{\cos 2\omega t}{3} + \ldots\right) & \left(\dfrac{\pi}{\omega} \leqslant t \leqslant \dfrac{2\pi}{\omega}\right) \end{cases} \tag{6.7}$$

into Equation (2.4).

The system given by Equation (6.6) can then be rewritten in the form $(0 < t < \pi/\omega)$

$$\alpha_1^{\cdot} - \frac{\nu^2}{u\cos\varphi}\beta = 0, \qquad \beta^{\cdot} + u\cos\varphi\alpha_1 + \Omega_0\gamma = f(t)\gamma$$

$$C\gamma^{\cdot\cdot} + Pl\gamma + \frac{Pl\,u\cos\varphi}{\nu^2}\Omega_0\alpha = \frac{Pl\,u\cos\varphi}{\nu^2}f(t)\,\alpha \tag{6.8}$$

where

$$\Omega_0 = \frac{2}{\pi}\mu\omega, \qquad f(t) = \frac{4}{\pi}\mu\omega\left(\frac{\cos 2\omega t}{3} + \frac{\cos 4\omega t}{15} + \ldots\right) \tag{6.9}$$

If we look upon the right-hand sides of the equations in (6.8) as known functions and apply the method of variation of arbitrary constants, we are led to a system of integral equations of the Volterra type, whose solutions can be obtained by the method of successive approximations.

The characteristic equation in first approximation will be

$$\lambda^4 + \frac{\nu^2}{C}\left(C + \frac{Pl}{\nu^2}\right)\lambda^2 + \frac{Pl}{C}(\nu^2 - \Omega_0^2) = 0 \qquad \left(0 \leqslant t \leqslant \frac{\pi}{\omega}\right) \tag{6.10}$$

and hence it follows that when $\Omega_0 > \nu$, i.e. when

$$v > \frac{\pi}{2}\frac{T}{T_0}Ru\cos\varphi \tag{6.11}$$

Equation (6.10) will have a positive root.

The condition given by Equation (6.11) has a similar structure to the inequality given by Equation (5.3) and is identical to Equation (4.12) of [3], which was obtained for a two-rotor gyrocompass. Further calculations, designed to obtain an analytical extension of the solutions to the interval $(\pi/\omega,\ T)$ and the setting-up of the characteristic equation, can be carried out as described in [3].

As in the case of the two-rotor compass, the characteristic equation of the system given by Equation (6.6) will be recurrent, i.e. of the form

$$\rho^4 + A_1\rho^3 + A_2\rho^2 + A_1\rho + 1 = 0 \tag{6.12}$$

and the stability regions are hence determined by the Liapunov inequalities

$$-2 < A_2 < 6, \; 4(A_2 - 2) < A_1^2 < \frac{1}{4}(A_2 + 2)^2 \tag{6.13}$$

7. In conclusion, we note that, as was shown in [3], the fulfilment of the condition (6.11) in the case of the two-rotor compass does not always lead to the instability of the system. This result is basically

related to the presence in the two-rotor gyrocompass of the gyroscopic moment

$$\Gamma = 2\,B \sin \varepsilon_0\, \delta^{\cdot} \tag{7.1}$$

which stabilizes the gyrosphere with respect to the generalized coordinate γ. In the case of a single-rotor compass, the moment given by Equation (7.1) is absent and the coordinate γ turns out to be unstabilized.

It follows that in the case of a single-rotor compass the fulfilment of condition (6.11) apparently leads to a general instability of the system. This is confirmed by numerical solutions of Equation (6.6) obtained with the aid of an electronic analog computer.

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